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Energy dissipation prediction in a line of colliding oscillators

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Abstract

In this paper the impact of a line of adjacent structures, or oscillators, is studied using an energy formulation. The energy exchange and dissipation from a collision of a pair of oscillators is studied by creating an equivalent oscillator pair, one has the energy of the in-phase motion and the other has the out-of-phase energy. It is found that the energy exchange between colliding oscillators is proportional to the initial kinetic energy difference of the oscillators and that work in the collision is proportional to the out-of-phase energy or difference energy. The kinetic energy at contact is then related to the mean oscillator energy, permitting a power balance equation to be written for each oscillator in line. The power balance equations have three independent variables for each pair of oscillators: the oscillator time averaged energies and the phase difference. This equation is run in a time-stepping procedure, with steps at the mean collision rate. The work in the collisions and internal oscillator dissipation is output as a function of time. A parameter study is conducted to see how the work changes with oscillator: separation, contact stiffness and contact damping.

1. Introduction

In a line of adjacent buildings damage can occur during an earthquake at high stress regions within the structure and at the local site of any collisions. Prediction of this damage must include the consideration that the precise properties of the structure, contact, and excitation are poorly defined, also the contact behavior itself almost certainly involves ill-defined non-linear damage mechanisms.

Previously collision problems have been successfully tackled by two different approaches. Elegant closed form analytical solutions for steady-state single frequency or multiple harmonic excitation have been found [1,2], by a solution termed the 'Periodic Greens Function.' This

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method is particularly appropriate to the field of machine dynamics where structural properties are well defined and excitation tends to be periodic. Alternatively for unsteady or random vibration the equations of motion and the momentum exchange relations can be solved numerically on a time-step basis [3,4]. Although this method is effective, stability of the second order differential equations requires quite a large number of points per second, for example 200/s in Ref. [4].

For the above methods collisions are described by a coefficient of restitution and contact duration that are independent of impact velocity, which is an acceptable assumption for heavy impacts and hard contact. These assumptions are not always safe, particularly in the case of interest here, as damage of a contact on a building will reduce impact velocities and also cause local softening. To accommodate these effects, expressions were derived in Ref. [5] for the contact duration and coefficient of restitution as a function of impact velocity, oscillator and contact properties, which are included in the method described here.

In this method it is suggested here that both types of structural damage could be assessed from an energy viewpoint by equating damage with work or dissipated energy. If there is a brittle fracture the local strain energy limit is exceeded. Alternatively if there is progressive damage as in reinforced concrete buildings the cumulative work from each load cycle or collision can be calculated. As this progressive failure is a non-linear process making exact solutions difficult, a simplified approach is employed with ensemble average input parameters and using the mean expected energy averaged over one cycle as the only dynamic parameter.

The basic assumption is that most of the energy is stored in a linear form, giving a single resonance frequency of a free oscillator. The effect of additional non-linear processes is to spread some energy away from the fundamental frequencies where it can no longer participate in the basic energy exchange and storage mechanism. This non-linear component can therefore be regarded as a loss and is assigned to the material loss factors. The non-linear collision mechanisms are therefore linearized in this analysis.

These input parameters and the energy variable are used in statistical energy analysis [6], the basis of which is that the power transferred between two linearly coupled oscillators, subjected to steady-state random vibration, is proportional to the difference between the mean oscillator energies. This approach has also been adapted for use with impulsive excitation to give responses in the time domain [7–9] as is also done here.

There are several advantages to operating with mean oscillator energy. First, the results are for ensemble averages and precise input data is not required. The second is that dissipated energy (or work) is a good objective measure of damage [10,11]. Therefore damage is a direct product of the calculations and is not a reduction from a more complicated procedure. The third is that energy or power calculations do not require linear behavior. Fourth, energy relations are simple for random excitation, where first order differential equations describe the time domain behavior. In a time-stepping procedure stability and accuracy do not require small time increments, and results are insensitive to small parameter variation.

In the work presented here a line of oscillators represents a line of adjacent buildings subjected to horizontal base excitation. The first objective is to set up a power balance equation for each oscillator. The second is to calculate the dissipated energy, as a function of time, from collisions and also internal hysteresis. The third is to see the effect of parameter changes on the energy dissipation.

The main element in the power balance equations, are the energy relations for collisions between a single pair of oscillators, which are derived by a new variation on the usual impact

dynamics approach. The first step was to calculate the coefficient of restitution for a collision between a pair of oscillators [5]. The next step was to make an energy transfer and loss equation for a pair of oscillators by combining the relative velocity equation and momentum equations. This equation can either be written in terms of the individual oscillator motions, or in terms of the relative and sum motions. This latter form is more convenient as the relative motion relates directly to energy loss in the collision. The time average power input to each oscillator was calculated from broadband random base excitation. A power balance was set up equating the power input from the base to the rate of energy dissipation in the collisions and in oscillator internal hysteresis. This first order differential equation is solved at each collision or at twice the mean cycle rate using a time-step procedure. All the parameters can be updated for each time step to account for damage dependent properties. The particular case of five adjacent oscillators is considered as an example and the effect of oscillator position, spacing, contact stiffness and damping are investigated.

2. Equivalent kinematic and energy models

Fig. 1 shows part of a line of oscillators spaced by a distance Δ_r. Each oscillator represents the fundamental transverse mode of a single building. The oscillator has a modal mass m_r and complex stiffness $k_r = k'_r(1 + i\eta_r)$, that includes a hysteretic loss factor η_r. The loss factor may be a function of frequency or amplitude, and so some thought should be given to the possible physical loss mechanisms present in the chosen application. The rth oscillator has instantaneous displacement and velocity u_r, v_r. The energy dissipating contact is described by a complex stiffness k_{cr} that includes a hysteretic loss factor η_{cr}. The dynamics of a single collision was modelled in Ref. [5] giving the 'coefficient of restitution' ε_r and contact duration t_{cr} as a function of the mechanical parameters and the ' impact ratio' β_r.

In Fig. 2 the same oscillator line is also represented in the energy notation used here. Vibrational power P_r is input from the base maintaining oscillator energy E_r and causing the 'difference energy' $\hat{E}_r$ of a single collision between oscillators r and $(r+1)$. The energy transferred between oscillators δE_r and absorbed $\hat{D}_r$, in this collision, is given in terms of the oscillator energy E_r. D_r is the dissipated energy from one load cycle within oscillator r.

Equating the power input to the four outputs of the rth oscillator, yields the fundamental power balance, that is the subject of this paper:

$$P_r = \dot{D}_r + \frac{\mathrm{d}E_r}{\mathrm{d}t} + \frac{(\delta E_r - \delta E_{r-1})}{(t_r + t_{r-1})}. \tag{1}$$

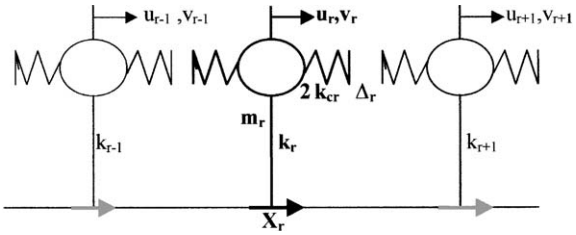

Fig. 1. Colliding oscillators from horizontal base excitation.

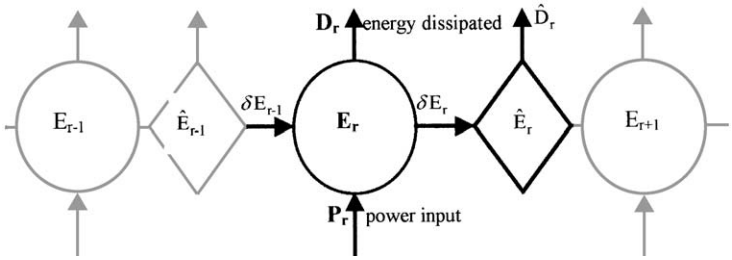

Fig. 2. Colliding oscillators using an energy description.

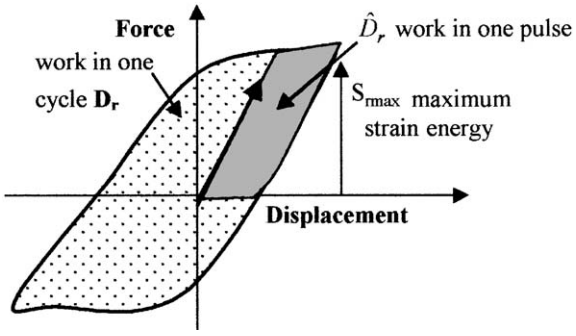

Fig. 3. Hysteresis loop for a yielding structural element.

The time averaged power input P_r from random excitation at the base, can be described in terms of the power spectral density of the base acceleration $\ddot{X}(f_r)$ at the natural frequency f_r [6]:

$$P_r = m_r \ddot{X}(f_r)/4. \tag{2}$$

This is an important formula as it shows that for a uniform spectral density the input power is only determined by the mass m_r. The vibration levels therefore increase until this power is dissipated either by damage or some other controlled damping mechanism.

The four energy terms that balance the input power are described from left to right. The first is the power lost through internal hysteresis or damage $\dot{D}_r$. If no contact is made this is controlled by the loss factor η_r and the angular frequency in rad/s $\omega_r = 2\pi f_r$:

$$\dot{D}_r = \eta_r \omega_r E_r. \tag{3a}$$

For yielding structural systems the loss factor is an amplitude dependent process rather than the frequency dependent viscous damping used in a linear analysis [12]. If this is true then loss factors measured at any loading rate can be used provided the correct amplitudes are employed. The main feature of loss factor is that it relates work to the maximum stress, which is likely to be the material yield stress. For harmonic excitation at a single frequency the loss factor can be obtained by measurement on a hysteresis loop seen in Fig. 3, using the definition:

$$\eta_r = D_r/(2\pi S_{r\,\max}) \tag{3b}$$

in which D_r is the area or dissipated work in one cycle, and $S_{r\,\max}$ is the maximum strain energy. This process replaces the non-linear system with an equivalent linear system dissipating the same work over one cycle. The loss factors of structural elements during progressive damage from cyclic loading tests are a function of displacement and number of cycles, and could be included in the time-step program. For a half sine pulse the hysteresis curve begins from the zero giving one quarter of the swept area of a cycle of harmonic loading, as seen in Fig. 3. The work is therefore $\hat{D}_r = \eta_r \pi S_{r\,\max}/2$. It should be noted that a linear viscous damping model with a half cycle impulse gives, as would be expected, half the work of a full cycle: $\hat{D}_r = \eta_r \pi S_{r\,\max}$.

The second term in Eq. (1) is the rate of increase of oscillator energy. The third term is the rate of energy transfer from oscillator r to $(r+1)$. The fourth is the rate of energy input from oscillator $(r-1)$ to oscillator r. $\delta E_r, \delta E_{r-1}$ are the respective energies transferred in a single collision, t_r, t_{r-1} are the times for oscillator r to make a half cycle including a collision with oscillator $(r+1)$ and $(r-1)$. The time for a complete cycle with two collisions is $(t_r + t_{r-1})$.

3. Energy relationships outside and within the collision

The calculation of the energy transferred and dissipated in a single collision is performed in two stages. First the free motion and energy of a pair of oscillators E_r, E_{r+1} is expressed as an equivalent pair of oscillators with the same total energy. One equivalent oscillator describes the sum of the instantaneous motions having the 'sum energy' of the pair $\bar{E}_r$, which is not directly influenced by the collision. The other equivalent oscillator describes the relative motion of the oscillator pair having the 'difference energy' or 'collision energy' $\hat{E}_r$, which is directly involved in the collision.

The second stage considers the dynamics during contact. This uses the 'coefficient of restitution' ε_r, and the law of linear momentum conservation to couple the two equivalent oscillators during contact. This coupling between the equivalent oscillators depends on the 'mass ratio,' μ_r. The transferred energy in a collision δE_r is obtained in terms of the oscillator kinetic energies T_r, T_{r+1} at the initial moment of impact. Finally, these instantaneous kinetic energies are related to the oscillator energies averaged over one cycle E_r, E_{r+1}. The expressions for the transferred and dissipated energy are, in the final form, a function of the three energy terms seen in Fig. 2, i.e. the oscillator energies E_r, E_{r+1} and the difference energy $\hat{E}_r$.

3.1. The dynamics of a free system

The two oscillators, r and $(r+1)$ in Fig. 1 are excited at the base by the same broadband random signal. This will give correlated displacements in each oscillator with each vibrating with a narrow random signal, predominately at its own natural frequency, ω_r and ω_{r+1} rad/s, with peak amplitudes U_r and U_{r+1}. After collisions begin only half of one cycle need be modelled so the usual free vibration representation is acceptable. The velocities and displacements are u_r, $u_r + 1$ at time t, relative to the arbitrary instant of contact $t = 0$:

$$u_r = A\mathrm{e}^{\mathrm{i}\omega_r t} + A*\mathrm{e}^{-\mathrm{i}\omega_r+1^t}, \quad u_{r+1} = B\mathrm{e}^{\mathrm{i}\omega_r t} + B*\mathrm{e}^{-\mathrm{i}w_r+1^t}, \tag{4a, b}$$

A, B are constants describing the displacements at contact, *denotes the complex conjugate. The displacements at time $t = 0$, are from Eq. (4) are therefore, $2\mathrm{Re}\{A\}$ and $2\mathrm{Re}\{B\}$ where Re denotes the real part. These displacements can be written in terms of the peak oscillator displacements U_r and U_{r+1} as

$$2A = U_r e^{i\theta_r}, \quad 2B = U_{r+1} e^{i\theta r+1} \tag{5a, b}$$

are the phase angles at time $t = 0$. The displacements and velocities at $t = 0$ from Eq. (5) are:

$$u_r = U_r \cos\theta_r, \quad u_{r+1} = U_{r+1} \cos\theta_{r+1}, \tag{6a, b}$$

$$v_r = -\omega_r U_r \sin\theta_r, \quad v_{r+1} = -\omega_{r+1} U_{r+1} \sin\theta_{r+1}. \tag{7a, b}$$

The free instantaneous oscillator displacements u_r, u_{r+1} and velocities v_r, v_{r+1} are related to the sum motions $\bar{u}_r, \bar{v}_r$ and difference motions $\hat{u}_r, \hat{v}_r$ of the equivalent sum-difference oscillator pair:

$$\bar{u}_r, \hat{u}_r = u_r \pm u_{r+1}, \quad \bar{v}_r, \hat{v}_r = v_r \pm v_{r+1}. \tag{8a, b}$$

Sum and difference masses $\bar{m}_r, \hat{m}_r$ and real stiffness' $\bar{k}_r, \hat{k}_r$ can be defined, and the 'mass ratio' μ_r:

$$\bar{m}_r, \hat{m}_r = m_r \pm m_{r+1}, \quad \bar{k}_r, \hat{k}_r = \mathrm{Re}(k_r \pm k_{r+1}), \quad \mu_r = \hat{m}_r/\bar{m}_r. \tag{9a, b, c}$$

The 'mass ratio' defines the coupling between the equivalent oscillator pair, as seen in the momentum equation (44). If, in the ideal case, oscillators r and $(r + 1)$ are identical, the mass ratio is zero and there is no coupling between the equivalent oscillator pair.

Although it is not essential for the analysis the 'mass ratio' can for some cases be described in terms of the scaling factor between two similar structures. If two oscillators are similar in geometry and material, as may well be the case for adjacent buildings, the scaling factor $s_r = l_{r+1}/l_r$, relating the characteristic lengths is also related to the masses, stiffness', natural frequencies and 'mass ratio.' For example if the structures are composed of rods of standard section acting in shear or extension, then the mass increases with length while the stiffness decreases with length:

$$m_{r+1} = s_r m_r, \quad k_{r+1} = k_{r/Sr}, \text{ and } \omega_{r+1} = \omega_r/S_r. \tag{10a, b, c}$$

By using the definitions of Eqs. (9) and (10) the mass and stiffness ratios become:

$$\frac{\hat{k}_r}{\bar{k}_r} = -\mu_r, \quad \frac{\hat{m}_r}{\bar{m}_r} = \mu_r, \quad \mu_r = \frac{(1 - s_r)}{(1 + s_r)}. \tag{11a, b, c}$$

The mass ratio μ_r is always less than unity, occurring for greatly dissimilar oscillators. However even if the scaling factor is as large as 2, the mass ratio of $-\frac{1}{3}$ suggests that the sum and difference oscillators are still not very strongly coupled.

In a collision between two free bodies only a certain proportion of the total mass is involved in the energy exchange and dissipation, this quantity is described here and in Ref. [5] as the 'interaction mass' m_{ir}. For the collision of oscillators an 'interaction stiffness' k_{ir} and 'interaction frequency' ω_{ir} are also defined:

$$m_{ir} = \frac{m_r m_{r+1}}{(m_r + m_{r+1})}, \quad k_{ir} = \mathrm{Re}\left(\frac{k_r k_{r+1}}{k_r + k_{r+1}}\right), \quad \omega_{ir} = \sqrt{\frac{k_{ir}}{m_{ir}}}. \tag{12a, b, c}$$

If the scaling relationships in Eq. (11) are applied, then the interaction frequency in Eq. (12) becomes the same as the sum motion frequency $\bar{\omega}_r$:

$$\omega_{ir}^2 = \bar{\omega}_r^2 = \bar{k}_r/\bar{m}_r. \tag{13}$$

In the approximate approach presented here the sum frequency or interaction frequency are regarded as being almost the same, their significance is that the energy of the oscillator pair is predominately stored in this frequency region. If the scaling definitions in Eq. (11) are employed the real parts of oscillator natural frequencies become:

$$\omega_r^2 = \bar{\omega}_r^2\frac{(1-\mu_r)}{(1+\mu_r)}, \quad \omega_{r+1}^2 = \bar{\omega}_r^2\frac{(1+\mu_r)}{(1-\mu_r)}. \tag{14a, b}$$

The sum frequency is therefore the geometric mean of the oscillator frequencies:

$$\bar{\omega}_r^2 = \omega_r\omega_{r+1}. \tag{15}$$

The energies at time t of the two oscillators E_r, E_{r+1} can be written as the sum of the strain energies, S_r, S_{r+1} and kinetic energy, T_r, T_{r+1}, i.e.

$$E_r = S_r + T_r, \quad E_{r+1} = S_{r+1} + T_{r+1}, \tag{16a, b}$$

where

$$T_r = \frac{m_r}{2}v_r^2, \quad T_{r+1} = \frac{m_{r+1}}{2}v_{r+1}^2, \quad S_r = \frac{k'_r}{2}u_r^2, \quad S_{r+1} = \frac{k'_{r+1}}{2}u_{r+1}^2.$$

The total kinetic energy and the difference in kinetic energies for the pair are T and ΔT:

$$T, \Delta T = T_r \pm T_{r+1}. \tag{17a, b}$$

By substituting from Eqs. (8) and (9) into Eq. (17) these are written in sum and difference form:

$$T = \frac{\bar{m}_r}{8}(\bar{v}_r^2 + \hat{v}_r^2) + \frac{\hat{m}_r}{4}\bar{v}_r\hat{v}_r, \quad \Delta T = \frac{\hat{m}_r}{8}(\bar{v}_r^2 + \hat{v}_r^2) + \frac{\bar{m}_r}{4}\bar{v}_r\hat{v}_r. \tag{18a, b}$$

The total strain energy, S, and strain energy difference, ΔS, are likewise:

$$S, \Delta S = S_r \pm S_{r+1}. \tag{19a, b}$$

These in sum and difference form are:

$$S = \frac{\bar{k}_r}{8}(\bar{u}_r^2 + \hat{u}_r^2) + \frac{\hat{k}_r}{4}\bar{u}_r\hat{u}_r, \quad \Delta S = \frac{\hat{k}_r}{8}(\bar{u}_r^2 + \hat{u}_r^2) + \frac{\bar{k}_r}{4}\bar{u}_r\hat{u}_r. \tag{20a, b}$$

The difference between the oscillator energies ΔS and ΔT that are given in Eqs. (18b) and (20b) will be seen in Section 3.3 to close to the energy transferred between oscillators in a collision.

The total energy terms T and S in Eqs. (18a) and (20a) have two parts, of which the first containing the sum mass or stiffness is always the larger. This is for two reasons. First, is that the ratios of the difference to sum terms in Eq. (11) is always less than unity, approximately equal to μ_r. Second, is that the quotient of the sum and difference displacements in Eqs. (18a) and (20a), i.e. $2\bar{v}_r\hat{v}_r/(\bar{v}_r^2 + \hat{v}_r^2)$ is also always less than unity. Furthermore, the average of this quotient for all possibilities of $\bar{v}_r$ and $\hat{v}_r$ is zero because of the implicit sine terms in the numerator.

If these secondary terms are therefore neglected in Eqs. (18a) and (20a) the total kinetic energy T and strain energy S can be approximately described by the sum and difference kinetic energies

and strain energies:

$$T = \bar{T}_r + \hat{T}_r, \quad S = \bar{S}_r + \hat{S}_r \tag{21a, b}$$

or:

$$T \cong \frac{\bar{m}_r}{8}(\bar{v}_r^2 + \hat{v}_r^2), \quad S \cong \frac{\bar{k}_r}{8}(\bar{u}_r^2 + \hat{u}_r^2). \tag{22a, b}$$

Two new and almost independent oscillators can now be defined. One has the energy of the sum motion $\bar{E}_r = \bar{T}_r + \bar{S}_r$, the other has the energy of the difference motion $\hat{E}_r = \hat{T}_r + \hat{S}_r$:

$$\bar{E}_r = \frac{\bar{m}_r}{8}\bar{v}_r^2 + \frac{\bar{k}_r}{8}u_r^2, \quad \hat{E}_r = \frac{\bar{m}_r}{8}\hat{v}_r^2 + \frac{\bar{k}_r}{8}\hat{u}_r^2. \tag{23a, b}$$

This simplification has zero error for identical oscillators, for completely dissimilar oscillators the error is zero on average, but has a possible 100% error for the worst case. If the restriction is made that $\frac{1}{2} < \omega_r/\omega_{r+1} < 2$ the maximum possible error is 33%, and zero on average, which is satisfactory for the application considered here.

3.2. The relationship between sum and difference energies

The energies of the equivalent oscillators have now been defined: the difference energy $\hat{E}_r$ that is involved in the collision; while the sum energy $\bar{E}_r$, describes the in-phase motion and is largely unaffected by the collision. The next step is to determine both of these energies as a function of time by referring back to the peak oscillator amplitudes U_r, U_{r+1} (or time average oscillator energies E_r, E_{r+1}) in Eqs. (6) and (7). This leads to the sum and difference energies at the reference time $t = 0$, written as a function of the 'difference phase' $\hat{\theta}_r$ describing the phase between the oscillator pair. This 'difference phase' is required along with the oscillator energies to define the 'difference energy.' Eqs. (21b) and (22b) for the sum and difference strain energies at the reference time $t = 0$ can be expanded using Eq. (6):

$$\bar{S}_r, \hat{S}_r = \frac{\bar{k}_r}{8}(U_r^2 \cos^2\theta_r + U_{r+1}^2 \cos^2\theta_{r+1} \pm 2U_r U_{r+1} \cos\theta_r \cos\theta_{r+1}). \tag{24a, b}$$

Therefore the total strain energy in Eq. (21b) at $t = 0$ is

$$S = \frac{\bar{k}_r}{4}(U_r^2 \cos^2\theta_r + U_{r+1}^2 \cos^2\theta_{r+1}). \tag{25}$$

Likewise using Eqs. (21a) and (22b) and (7), the sum and difference kinetic energies at $t = 0$, become

$$\bar{T}, \hat{T} = \frac{\bar{m}_r}{8}(\omega_r^2 U_r^2 \sin^2\theta_r + \omega_{r+1}^2 U_{r+1}^2 \sin\theta_{r+1}^2 \pm 2\omega_r\omega_{r+1} U_r U_{r+1} \sin\theta_r \sin\theta_{r+1}). \tag{26a, b}$$

The total kinetic energy at $t = 0$ is therefore given from Eqs. (21a) and (26):

$$T = \frac{\bar{m}_r}{4}(\omega_r^2 U_r^2 \sin^2\theta_r + \omega_{r+1}^2 U_{r+1}^2 \sin^2\theta_{r+1}). \tag{27}$$

By using Eqs. (13) and (14) to expand the frequency and mass terms, this may be written as

$$T = \frac{\bar{k}_r}{4(1-\mu_r^2)}(U_r^2 \sin^2\theta_r(1-\mu_r)^2 + U_{1+r}^2 \sin^2\theta_{1+r}(1+\mu_r)^2). \tag{28}$$

The terms containing μ_r^2 can be neglected for similar oscillators because they are small. Furthermore the opposing sign in $2\mu_r$ for the two components will tend to cancel, leaving no dependence in μ_r. The total energy at the reference time is $E = T + S$; which from Eqs. (25) and (28) becomes

$$E = \frac{\bar{k}_r}{4}(U_r^2 + U_{r+1}^2). \tag{29}$$

This is dependent only upon the peak velocity or displacement and the sum stiffness (or mass), but is invariant with the phase at the reference time. Eq. (29) is of course only perfectly accurate when the oscillators are identical. However, the significance of this equation for total energy is that it provides some verification for the sum and difference energy expressions used in its derivation, indicating that the physics has not entirely drowned in the approximations.

The difference energy $\hat{E}_r$ and sum energy $\hat{E}_r$, are also invariant over the cycle if there is no exchange or loss of energy and are given from Eqs. (25) and (27) using the strain energies and kinetic energies,

$$\bar{E}_r = \bar{T}_r + \bar{S}_r, \quad \hat{E}_r = \hat{T}_r + \hat{S}_r, \tag{30a, b}$$

where

$$\bar{E}_r, \hat{E}_r = \frac{E}{2}(1 \pm q_r \cos(\hat{\theta}_r)), \tag{31a, b}$$

E is the total energy given in Eq. (29), or more accurately the sum of oscillator energies: $(E_r + E_{r+1})$. E is the maximum for the sum or difference energy. The multiplier for the phase dependent part q_r is a function of the oscillator amplitudes, and is unity for equal amplitudes:

$$q_r = \frac{2U_rU_{r+1}}{(U_r^2 + U_{r+1}^2)}.$$

Alternatively this could be written in terms of oscillator energies:

$$q_r = \frac{2r}{(1+r^2)}, \tag{32}$$

where $r = \sqrt{(E_r k'_{r+1})/(E_{r+1}k'_r)}$ or $\sqrt{(E_r\omega_{r+1}^2 m_{r+1})/(E_{r+1}\omega_r^2 m_r)}$.

The sum and difference energies in Eq. (31) are plotted in Fig. 4 against the 'difference phase': $\hat{\theta}_r$. These energy quantities are slowly changing functions dependent upon the 'difference phase' $\hat{\theta}_r$, the product of the difference in oscillator natural frequencies $\hat{\omega}_r$ and preceding time duration t:

$$\hat{\theta}_r = \hat{\omega}_r t + \hat{\theta}_{0r}, \tag{33}$$

where the difference frequency is

$$\hat{\omega}_r = \omega_r - \omega_{r+1}.$$

As the excitation comes from a travelling wave at the base there is an initial difference phase $\hat{\theta}_{0r}$ between oscillator pairs. The smooth envelope of the difference energy is seen also in Fig. 4 to be

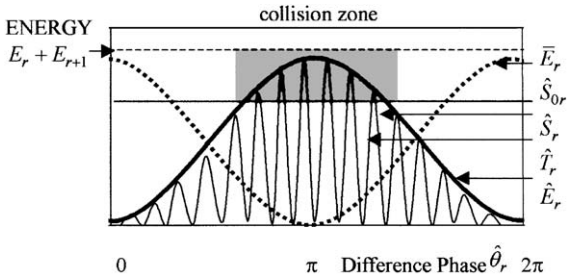

Fig. 4. Sum and difference energy as function of difference phase.

the sum of the rapidly fluctuating 'difference strain energy' $\hat{S}_r$ and 'difference kinetic energy' $\hat{T}_r$ arising from the relative motion of the oscillators, at any instant of time. This relative motion occurs at the 'sum frequency' $\bar{\omega}_r$ defined in Eq. (13), and the energies in Fig. 4 fluctuate at $2\bar{\omega}_r$. The sum frequency is almost identical to the 'interaction frequency' ω_{ir} defined in Eq. (12).

The 'difference phase' in Eq. (33) is almost zero at the start of excitation as $t = 0$; the two oscillators move in phase and there is no possibility of collision. Alternatively from Eq. (31b) it could be said that the difference energy is almost zero. As excitation persists the difference phase in Eq. (33) increases causing the difference energy to climb the hill in Fig. 4 until it reaches what is termed here the 'strain energy threshold' $\hat{S}_{0r}$:

$$\hat{S}_{0r} = k_{ir}\Delta_r^2/2 \tag{34}$$

at which point the difference displacement $\hat{u}_r$ is equal to the oscillator spacing Δ_r. Collisions occur after this point, with a strength that depends on the 'impact ratio' β_r defining the ratio between the difference kinetic energy at initial contact $\hat{T}_r$ and the strain energy threshold:

$$\beta_r^2 = \hat{T}_r/\hat{S}_{0r}. \tag{35}$$

The energy available for damage is the difference kinetic energy at impact, which from Eq. (30b) is

$$\hat{T}_r = \hat{E}_r - \hat{S}_{0r}. \tag{36}$$

This is seen as the shaded area in Fig. 4. The difference kinetic energy in this collision band will be referred to as the 'collision energy.'

3.3. The system dynamics during contact

The previous sections described the oscillator dynamics before and up to contact, while here the period of contact is considered. These two regions may be compared using Ref. [5], where it is shown that the only change is the additional presence of the contact stiffness k_{cr} changing the system 'interaction frequency' ω_{ir}, and the 'impact ratio' β_r from their uncoupled values.

This section has two parts. First a calculation of the collision rate between two oscillators is made, giving the time period for one half cycle t_r which is required in the power balance in Eq. (1). This is as a function of the interaction frequency in and out of contact. The second part uses the coefficient of restitution and the conservation of momentum to express a single collision between two oscillators in energy terms and relating this to the kinetic energy at contact.

3.3.1. The relationship between contact time, impact strength and collision rate

The presence of the contact stiffness increases the 'impact ratio' β_r in Eq. (35) to what is now termed the 'impact strength' $\hat{\beta}_r$. This was shown in Ref. [5] to be

$$\hat{\beta}_r = \beta_r \sqrt{1 + k'_{cr}/k_{ir}}. \tag{37}$$

The 'interaction frequency' ω_{ir} in Eq. (12c) becomes modified by the contact stiffness to $\hat{\omega}_{ir}$, which defines the maximum contact time $\hat{t}_{cr}$:

$$\hat{\omega}_{ir} = \omega_{ir} \sqrt{1 + k'_{cr}/k_{ir}}, \quad \hat{t}_{cr} = \pi/(\hat{\omega}_{ir}). \tag{38a, b}$$

The contact time t_{cr} increases with the 'impact strength,' but achieves the maximum $\hat{t}_{cr}$, corresponding to the free body collision value when $\hat{\beta}_r \gg 1$. In general the contact time is [5]

$$t_{cr} = \hat{t}_{cr} \sqrt{1 - 1/\sqrt{1 + \hat{\beta}_r^2}}. \tag{39a, b}$$

This contact time is illustrated in Figs. 5a and b, also seen is the time t_r of a half cycle of the difference displacement. Fig. 5a gives the difference strain energy as the 'interaction phase' changes through 2π or one cycle of the difference motion. Outside of contact this occurs at the sum frequency $\bar{\omega}_r$ given in Eq. (13). The change in difference strain energy with interaction phase, are also seen as the fluctuations under the envelope of the difference energy in Fig. 4. The interval $0 - \pi$ in Fig. 5a includes one collision when the difference strain energy exceeds the strain energy threshold. This is related in Eq. (34) to the oscillator spacing.

The time for one half cycle t_r can be calculated from this phase representation as the total interaction phase change of π is equal to the sum of the phase change in and out of contact:

$$\pi = \omega_{ir}(t_r - t_{cr}) + \hat{\omega}_{ir} t_{cr}. \tag{40}$$

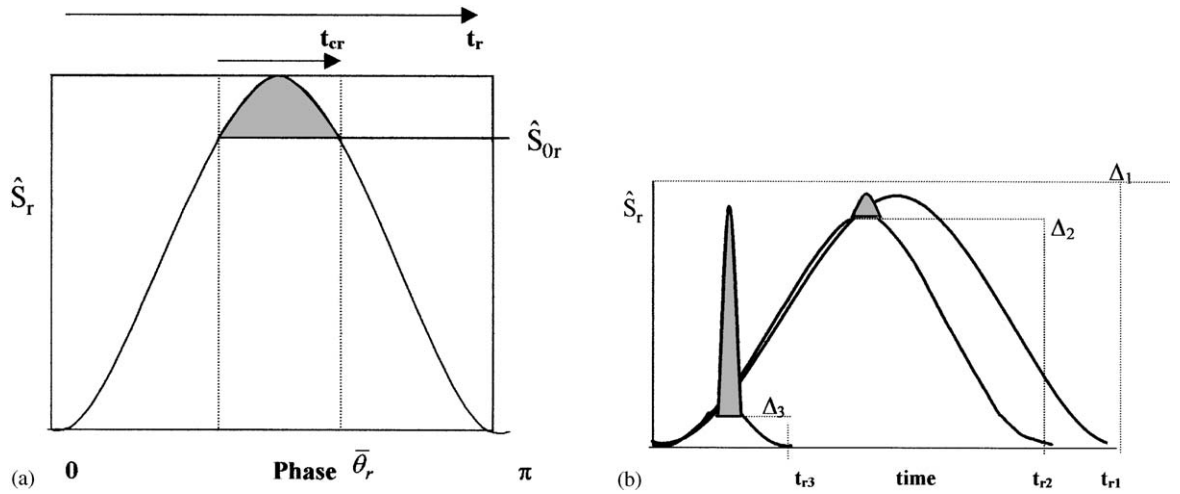

Fig. 5. (a) Difference strain energy as a function of interaction phase $\bar{\theta}_r$. (b) Half cycle time of difference strain energy as a function of oscillator spacings Δ.

Rearrangement gives

$$t_r = \frac{\pi}{\omega_{ir}}\left(1 - \frac{t_{cr}}{\pi}(\hat{\omega}_{ir} - \omega_{ir})\right). \tag{41}$$

The result can be interpreted with reference to Fig. 5b, where the difference strain energy is given as a function of time for three different oscillator separations. The first separation Δ_1 exceeds the difference displacement and there is no collision and the half period takes the maximum of π/ω_{ir}. As the separation becomes smaller to Δ_2 and Δ_3 a larger proportion of the phase change is spent within contact thereby reducing the half-period time. This explains the observation that, the collision rate, $2/(t_r + t_{r-1})$, increases with decreasing separation or increasing excitation level.

3.3.2. Energy transferred and dissipated in a collision

When the oscillators are in collision a coefficient of restitution ε_r is applied, as with free body collision, to describe the attenuation of difference velocity from the start to the end of the contact $\hat{v}_r, \hat{v}'_r$:

$$\hat{v}'_r = -\varepsilon_r \hat{v}_r. \tag{42}$$

The coefficient of restitution is related in Ref. [5] to the imaginary part of the interaction frequency $\mathrm{im}(\hat{\omega}_{ir}) = \hat{\gamma}_{ir}$, and the contact time t_{cr} by

$$\varepsilon_r = \exp(-\hat{\gamma}_{ir} t_{cr}). \tag{43}$$

In Appendix A the interaction frequency is related to the loss factor η_{cr} of the contact stiffness. For linear viscous damping: $\hat{\gamma}_{ir} \cong (\eta_{cr}/2)\mathrm{Re}(\hat{\omega}_{ir})$. However for hysteretic damping mechanisms which may be non-linear $\hat{\gamma}_{ir} \cong (\eta_{cr}/4)\mathrm{Re}(\hat{\omega}_{ir})$. For both cases the loss factor is obtained from the cyclic test plot seen in Fig. 3.

The analysis of the contact also requires the equation for conservation of linear momentum, given here in terms of initial and final sum and difference velocity $\bar{v}_r, \bar{v}'_r$ and $\hat{v}_r, \hat{v}'_r$:

$$\bar{v}_r - \bar{v}'_r = \mu_r(\bar{v}'_r - \hat{v}_r). \tag{44}$$

The 'mass ratio' μ_r given in Eq. (11b) controls the coupling between the sum and difference motion during the collision. The energy increase of oscillator r and $(r+1)$ in the collision $\delta E_r, \delta E_{r+1}$ is due to the change in kinetic energy

$$\delta E_r = T'_r - T_r, \quad \delta E_{r+1} = T'_{r+1} - T_{r+1}. \tag{45a, b}$$

Eqs. (42),(44) and (45) can now be combined to give the energy transfer:

$$\delta E_r, \delta E_{r+1} = \mp \frac{m_{ir}}{2}(1 + \varepsilon_r)\hat{v}_r\left(\bar{v}_r + \frac{1}{2}\hat{v}_r(\mu_r(1 + \varepsilon_r) \pm (1 - \varepsilon_r))\right). \tag{46a, b}$$

If the coefficient of restitution $\bar{\varepsilon}_r \to 1$, as for the zero loss case, Eq. (46) becomes

$$\delta E_r, \delta E_{r+1} = m_{ir}\bar{v}_r\hat{v}_r + \mu_r\hat{v}_r^2. \tag{47}$$

As the interaction mass $m_{ir} \cong \bar{m}_r/4$ the initial kinetic energy difference ΔT in Eq. (18b) can be compared with the transferred energy in Eq. (47):

$$\frac{\Delta T}{\delta E_r} = \frac{1 + \mu_r\left(((\bar{v}_r^2 + \hat{v}_r^2)/2)/\bar{v}_r\hat{v}_r\right)}{\left(1 + \mu_r\left(\hat{v}_r^2/\bar{v}_r\hat{v}_r\right)\right)}. \tag{48}$$

It can be seen from this expression that for most practical cases the ratio of the two energy terms is close to unity. The first reason is that the mass ratio μ_r is less than unity. The second is that the average of the sum or difference velocities is the same. The greatest possible error occurs for totally dissimilar oscillators where $\mu_r = 1$, and if the sum velocity at collision is zero, then

$$\left.\frac{\Delta T}{\delta E_r}\right|_{\min} = \frac{1}{2}.$$

However as this study restricts the mass ratio to be less than 0.3 fractional errors of only 0.1 are expected. It is therefore an interesting result that the energy exchanged between two similar oscillators in a collision is on average close to the initial kinetic energy difference.

The sum and difference velocities defined in Eq. (8b) can be used to change Eq. (44) for energy transfer into the terms of oscillator kinetic energy and collision energy

$$\delta E_r, \delta E_{r+1} = \mp(1+\varepsilon_r)\left[m_{ir}\left(\frac{T_r}{m_r} - \frac{T_{r+1}}{m_{r+1}}\right) + \frac{\mu_r \hat{T}_r}{2}(1+\varepsilon_r)\right] - \frac{\hat{D}_r}{2}. \tag{49a, b}$$

where the dissipated energy in the collision $\hat{D}_r$, and the collision energy $\hat{T}_r$ are

$$\hat{D}_r = \hat{T}_r(1-\varepsilon_r^2), \quad \hat{T}_r = \frac{m_{ir}}{2}\hat{v}_r^2.$$

The energy transfer terms have three parts. As discussed above, for coefficients of restitution approaching unity, Eq. (49) is close to the initial kinetic energy difference ΔT in Eq. (18b). This also corresponds to the central ideas in statistical energy analysis. The second term in the brackets shows the energy is transferred preferentially from the greater to smaller mass, as indicated by the mass ratio μ_r defined in Eq. (9). The final term shows that the dissipated energy is subtracted equally from energy transferred in both directions. This dissipated energy expression, also found in Ref. [13], is the 'collision energy' modified by the coefficient of restitution.

3.4. Kinetic energy at contact and total energy

The transfer of energy between oscillator r and $(r+1)$ in a collision was shown above to be close to the kinetic difference in oscillator energy ΔT at the instant of contact. However for this result to be useful when applied to a pair or a series of colliding oscillators, the kinetic energy difference at the instant of contact must be related to the oscillator energy difference ΔE, which is the main task in this section. This is achieved using the average 'energy ratio' as there is insufficient information for a deterministic solution.

The 'energy ratio' $\Delta T/\Delta E$ expanded using the difference in oscillator kinetic energy ΔT and strain energy ΔS is

$$\frac{\Delta T}{\Delta E} = \frac{\Delta T}{(\Delta T + \Delta S)}. \tag{50}$$

Eqs. (18b) and (20b) give values for ΔT and ΔS, in which the simplified versions are used here that neglect the terms containing $\hat{k}_r$ and $\hat{m}_r$ as only the energy transfer term in Eq. (49) containing T_r and T_{r+1} is required, thus

$$\frac{\Delta T}{\Delta E} = \frac{\bar{m}_r \hat{v}_r \bar{v}_r}{(\bar{m}_r \bar{v}_r \hat{v}_r + \bar{k}_r \bar{u}_r \hat{u}_r)}. \tag{51}$$

Using Eq. (13), this may be written in non-dimensional parameters

$$\frac{\Delta T}{\Delta E} = \frac{\alpha_r \beta_r}{(\alpha_r \beta_r + 1)}. \tag{52}$$

β_r is already defined as the 'impact ratio' in Eq. (35) but is given in alternative form with α_r:

$$\alpha_r = \frac{\bar{v}_r}{\omega_{ir} \bar{u}_r}, \quad \beta_r = \frac{\hat{v}_r}{\omega_{ir} \hat{u}_r}.$$

At contact the difference displacement $\hat{u}_r = \Delta_r$, the oscillator spacing. The 'impact ratio' β_r is always positive at contact, as both the relative displacement and relative velocity are positive (i.e. moving towards each other). However the 'sum ratio' α_r has no specific value as the sum and difference energies are independent. Although it can be described as a harmonic velocity of the form $\sin(\omega_{ir} t + \bar{\theta}_{0r})$ divided by the corresponding harmonic displacement. If the contact occurs at $t = 0$ the 'sum ratio' becomes

$$\alpha_r = \frac{\sin \bar{\theta}_{0r}}{\cos \bar{\theta}_{0r}}, \quad -\pi > \bar{\theta}_{0r} > \pi. \tag{53}$$

Substitution of Eq. (53) into Eq. (52) gives

$$\frac{\Delta T}{\Delta E} = \frac{\beta_r \sin \bar{\theta}_{0r}}{(\cos \bar{\theta}_{0r} + \beta_r \sin \bar{\theta}_{0r})}. \tag{54}$$

It can be seen that $\Delta T/\Delta E$ tends to unity for large impact strength when $\beta_r \to \infty$, this means the collision occurs at the point of maximum velocity when the displacement is relatively small, the collision dynamics are then the same as for two free bodies.

There is however, the possibility that the sum ratio is negative for low impact strengths when $\beta_r < \cot \bar{\theta}_{0r}$. This is slightly disconcerting as in these instances the energy transferred in the collision is in the opposite direction of the oscillator energy difference ΔE. An example would be when one oscillator was momentarily stationary with strain energy but no kinetic energy. Any moving body that collides with it has more kinetic energy thus transferring energy to the stationary oscillator. This is irrespective of the fact that the first oscillator may have more total energy but in the form of strain energy.

However the intention here is to ignore particular events and work with ensemble averages so that some prediction of behavior can be made. The value of $\bar{\theta}_{0r}$ is unknown at impact, but it is assumed to have an equal possibility of taking any angle between $\pm\pi$ permitting a mean of Eq. (54) to be calculated, i.e.

$$\left\langle \frac{\Delta T}{\Delta E} \right\rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\beta_r \sin \bar{\theta}_{0r}}{(\cos \bar{\theta}_{0r} + \beta_r \sin \bar{\theta}_{0r})} \mathrm{d}\bar{\theta}_{0r}. \tag{55}$$

By making the substitutions that $\tan \delta = \beta_r$, and $\phi = \bar{\theta}_{0r} - \delta$, the integral becomes

$$\left\langle \frac{\Delta T}{\Delta E} \right\rangle = \frac{\beta_r}{2\pi \sqrt{1 + \beta_r^2}} \int_{\pi}^{\pi} \frac{\sin(\phi + \delta)}{\cos \phi} \mathrm{d}\phi. \tag{56}$$

If the numerator is then expanded using the sine sum formula, execution the integral yields

$$\left\langle \frac{\Delta T}{\Delta E} \right\rangle = \frac{\beta_r^2}{(1+\beta_r^2)} \tag{57}$$

or using Eqs. (35) and (36):

$$\left\langle \frac{\Delta T}{\Delta E} \right\rangle = \frac{(\hat{E}_r - \hat{S}_{0r})}{\hat{E}_r}, \tag{58}$$

which is plotted in Fig. 6.

Eq. (57) or Eq. (58) has a satisfyingly simple and plausible form. If $\hat{E}_r < \hat{S}_{0r}$, there is insufficient 'difference energy' for collisions to occur. When $\hat{E}_r = \hat{S}_{0r}$ the oscillators only touch without kinetic energy, so β_r is zero. For small impact strength when $\beta_r < 1$, only a fraction of the total oscillator energy is involved in the collision. Alternatively when $\beta_r \gg 1$, all the oscillator energy is involved in the collision, as for collisions between free bodies.

Eq. (49) can now be expressed in the desired form for the power balance in Eq. (1), i.e. in terms of the primary system variables $E_r, E_{r+1}, \hat{E}_r$, and parameters $\varepsilon_r, \hat{S}_{ro}, m_r, m_{r+1}$:

$$\delta E_r, \delta E_{r+1} = \mp(1+\varepsilon_r)\left[m_{ir}\left(1 - \frac{\hat{S}_{0r}}{\hat{E}_r}\right)\left(\frac{E_r}{m_r} - \frac{E_{r+1}}{m_{r+1}}\right) + \mu_r(\hat{E}_r - \hat{S}_{0r})\frac{(1+\varepsilon_r)}{2}\right] - \frac{\hat{D}_r}{2}. \tag{59a, b}$$

The parameters: $m_{ir}, \hat{D}_r, \mu_r$ are functions of the primary variables and parameters.

3.5. The energy absorbed by the oscillators in one cycle

In the series of base excited oscillators in Fig. 1 or Fig. 2 the input power is balanced by the power lost in the collisions and the oscillator internal hysteresis. Now that the dissipation in the collisions has now been defined, it now only remains to look more carefully at the rate of energy loss $\dot{D}_r$ within the oscillators, seen in Eq. (1). Before collisions occur this is controlled only by the internal hysteresis as described in Eq. (3). However during the collision some of the difference energy, namely $\hat{D}_r$, is lost and a component of this difference energy is stored not only in the contact stiffness but also in the oscillator 'interaction stiffness.' So a modified version of the loss from the oscillators is presented here that is valid during collision. The main argument is that the

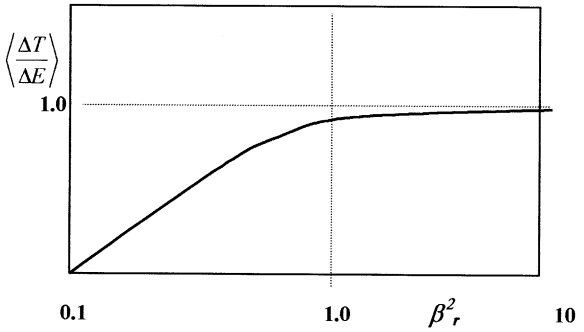

Fig. 6. Proportion of energy available in the collision as a function of the 'collision ratio' β_r^2.

oscillator energy E_r can be divided into sum and difference energy of the oscillator pairs r, $(r-1)$ and r, $(r+1)$:

$$E_r = \frac{(\bar{E}_r + \bar{E}_{r-1})}{4} + \frac{(\hat{E}_r + \hat{E}_{r-1})}{4}. \tag{60}$$

It is also assumed that only the difference energy is affected by the collision.

The two parts of the dissipated power $\dot{D}_r$ within oscillator r during collision with the oscillators either side are

$$\dot{D}_r = \eta_r \omega_r \frac{(\bar{E}_r + \bar{E}_{r-1})}{4} + \frac{1}{(t_r + t_{r-1})} \left(\begin{array}{c} \left(\dfrac{\hat{D}_r}{2} \left(\dfrac{\eta_r k_{ir}}{\eta_r k_{ir} + \eta_{cr} k'_{cr}} \right) + \pi \eta_r \dfrac{\hat{S}_{0r}}{2} \right) \\ + \left(\dfrac{\hat{D}_{r-1}}{2} \left(\dfrac{\eta_{r-1} k_{ir-1}}{\eta_{r-1} k_{ir-1} + \eta_{cr-1} k'_{cr-1}} \right) + \pi \eta_{r-1} \dfrac{\hat{S}_{0r-1}}{2} \right) \end{array} \right). \tag{61}$$

The first term is the power dissipated in the sum motion. The time for two half cycles including a collision on either side of oscillator r is in the denominator of the second term. This time period is given in Eq. (41). The next term is the proportion of the collision energy $\hat{D}_r$ that is dissipated in the oscillators. This expression is taken from equation (A.10). The following term in the same brackets is the proportion of the difference strain energy dissipation before the collision. The same two quantities follow for the relative motion between oscillator r and $(r-1)$.

4. The time-step program

All the parameters in the power balance of Eq. (1) are now established and it is possible to run it in a time-step program from zero time for each of the n oscillators in a line. Eq. (1) for oscillator r can be rearranged to give the oscillator energy increment $\mathrm{d}E_r$ due to a small time step $\mathrm{d}t$:

$$\mathrm{d}E_r = \mathrm{d}t \left(P_r - \dot{D}_r - \frac{(\delta E_r - \delta E_{r-1})}{(t_r + t_{r-1})} \right). \tag{62}$$

Apart from the input power P_r the other terms are functions of three independent variables for each oscillator pair, these are the oscillator energies E_r, E_{r+1} and the difference phase $\hat{\theta}_r$. The difference energy $\hat{E}_r$, the coefficient of restitution ε_r and the half cycle period t_r are found from these three variables.

The first step is to define the oscillator energies and difference phase at zero time. An increment in time $\mathrm{d}t$ is made causing the oscillator increment $\mathrm{d}E_r$ in accordance with Eq. (62), involving Eqs. (1), (59) and (61) for energies $\delta E_r, \delta E_{r+1}$ and $\hat{E}_r$. The oscillator energies E_r, E_{r+1} and difference phase $\hat{\theta}_r$, are increased by the increments $\mathrm{d}E_r, \mathrm{d}E_{r+1}$ and $\mathrm{d}\hat{\theta}_r$, then the procedure is repeated.

The procedure to calculate the increment in difference phase $\mathrm{d}\hat{\theta}_r$ begins with the difference phase taken from Eqs. (29) and (30):

$$\cos \hat{\theta}_r = \frac{(E_r + E_{r+1} - 2\hat{E}_r)}{2\sqrt{E_r E_{r+1}}}. \tag{63}$$

The change in the cosine of the difference phase is caused by changes in the oscillator energies, the collision energy and the time between calculations, dt:

$$\mathrm{d}(\cos\hat{\theta}_r) = \frac{\partial\cos\hat{\theta}_r}{\partial E_r}\mathrm{d}E_r + \frac{\partial\cos\hat{\theta}_r}{\partial E_{r+1}}\mathrm{d}E_{r+1} + \frac{\partial\cos\hat{\theta}_r}{\partial\hat{E}_r}\mathrm{d}\hat{E}_r + \cos(\hat{\theta}_r + 2\hat{\omega}_r\mathrm{d}t) - \cos(\hat{\theta}_r). \tag{64}$$

The last two terms represent the change due to the time dt and is taken from Eq. (33). Application of Eqs. (63) and (64) gives

$$\mathrm{d}(\cos\hat{\theta}_r) = \frac{1}{4\sqrt{E_rE_{r+1}}} \times \left(\frac{\mathrm{d}E_r}{E_r}(E_r - E_{r+1} + 2\hat{E}_r) + \frac{\mathrm{d}E_{r+1}}{E_{r+1}}(E_{r+1} - E_r + 2\hat{E}_r) - 4\mathrm{d}\hat{E}_r\right) + \cos(\hat{\theta}_r + 2\hat{\omega}_r\mathrm{d}t) - \cos(\hat{\theta}_r). \tag{65}$$

The oscillator energy increments $\mathrm{d}E_r$ and $\mathrm{d}E_{r+1}$ are given in Eq. (62). The change in collision energy $\mathrm{d}\hat{E}_r$ is the work of the collision: $\mathrm{d}\hat{E}_r = -\hat{D}_r$, found in Eq. (49).

The calculations should ideally be made at time intervals dt corresponding to the interval t_r, for each oscillator, to give the energy available for the next collision. However the collision rate for each oscillator is different, and so this is not possible, as the calculation method requires time steps common to all the oscillators. The result will be slightly sensitive to the selected step size, as Eq. (39) and hence Eq. (41) is non-linear for light contact. Therefore the average collision period t_r for the n oscillators was selected here as it is the best compromise.

5. Test of the time-step program

A series of tests were conducted to check the correct running of the time-step program and to examine the expected effect of parameter changes on the work of collision, and the internal energy loss of the oscillators. Five adjacent oscillators were used in the scheme shown in Fig. 1. The oscillator masses are identical: $m_r = 1$ kg. The oscillator spacing is identical: $\Delta_r = 0.1$ m. A wave approaches from the left at a speed of $c = 1$ m/s. As the spacing between oscillator r and $(r+1)$ is Δ_r the initial difference phase for collision r in Eq. (33) is

$$\hat{\theta}_r = \frac{\omega_r c}{\Delta_r}. \tag{66}$$

For the reference the contact stiffness is constant $k_{cr} = 100$ N/m. The contact loss factor is also constant $\eta_{cr} = 1$. The oscillator loss factors are $\eta_r = 0.1$. The oscillator stiffness' N/m are: $k_1 = 10$, $k_2 = 14$, $k_3 = 12$, $k_4 = 10$, $k_5 = 14$.

5.1. The reference test

The first test was intended to check the stability of the program and provide a reference for a small parameter study. A constant base excitation level of $\ddot{X} = 0.1(\mathrm{ms}^{-2})^2/\mathrm{Hz}$ between 0 and 5 Hz was applied. The natural frequencies were arranged to be at about 0.5 Hz. The program steps the

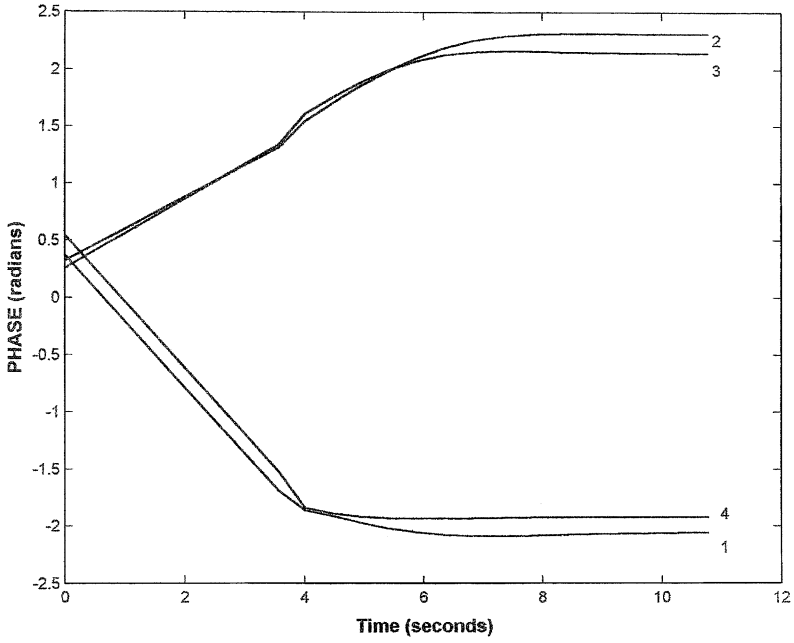

Fig. 7. Difference phase, separation 0.1 m, contact stiffness 100(1 + i)N/m.

time automatically at the mean collision frequency, and adjusts to increase with changes due to amplitude; 20 time steps gave 12 s of response in this case. The collision phase is shown in Fig. 7 for the four pairs. At zero time the starting phase is approximately 0.3 rad for each collision as determined by Eq. (66). The phase steps forward as described by Eq. (33) until the first collision occurs after about 3 s, at this moment the difference strain energy $\hat{S}_r$ is equal to the collision strain energy threshold $\hat{S}_{0r}$. It is interesting that the sum of the difference phases remains almost a constant, only offset from zero by the wave time delay. This fixed phase relationship suggests a single mode of vibration. This is rather deceptive as there are, in fact, five possible vibration modes, and this result is just a product of the method representing some average behavior.

In the remaining figures collision energy between a pair of oscillators has bold labels while the oscillator energy has normal labels. In Fig. 8 the difference energy, from Eq. (31), increases with increasing oscillator energies and difference phase until the first collision. The energy available for damage is the difference between difference energy $\hat{E}_r$ and the strain energy threshold. For collision 1 the strain energy threshold calculated from Eq. (34) is 0.028 J, which is marked in Fig. 8. Stability seems to occur very rapidly after about two collisions, probably because this is only a first order differential power balance equation. The difference energy is greatest for the outer two collisions 1,4 and also the outer two oscillator energies 1,5. This is also the observation from 'pounding' during an earthquake within a line of colliding buildings, where the outer pairs are most damaged [3]. This is because the outer oscillators have greater freedom than the other oscillators in having only one collision/cycle rather than two. Although the collision rate is lower for the outer oscillators, the strength of collision is greater because of the greater freedom of

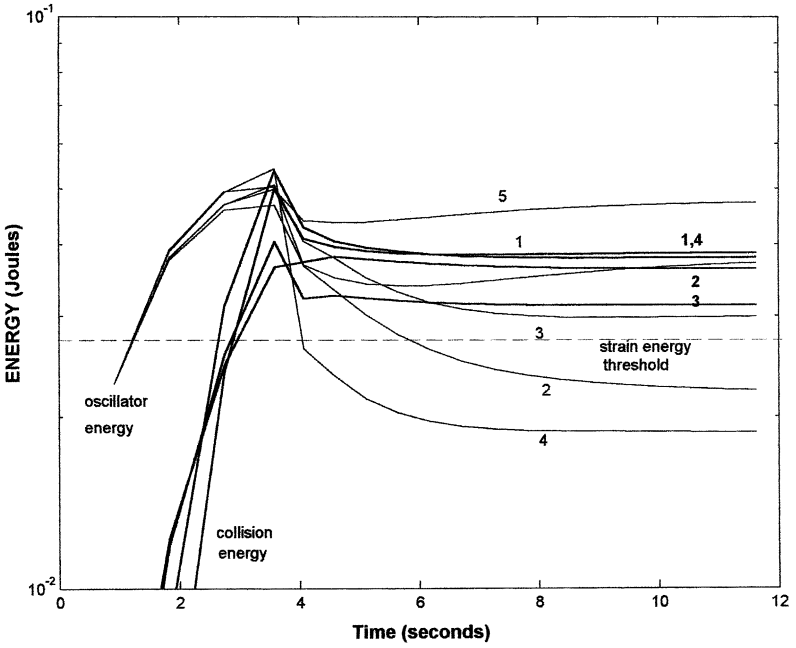

Fig. 8. Oscillator energy — and collision energy ——, separation 0.1 m, contact stiffness 100(1 + i)N/m.

movement. This seems to be the general principle of collision damage, i.e. it increases with the permitted freedom of movement.

The collision work is simply the time integral of the power and is shown in Fig. 9 to have the same trend. The work increases linearly with time as the input power, and hence dissipated power, is constant. A simple power balance calculation showed that the power input matched the sum of dissipation from internal hysteresis and the collisions, indicating that broadly speaking at least, the program is operating properly. It also demonstrates that the total work is rather easily predicted, and only the distribution of this work is determined by the mechanical parameters.

The internal oscillator work begins from zero time, it takes a slight decrease in rate when collisions begin after 3.5 s. The oscillator energy in Fig. 8 is at least twice the difference energy, leading to the observation from Fig. 9 that the internal work is always greater than the collision work by the same factor. The collision work is however often more significant from the damage viewpoint as it is local to the region of contact.

5.2. *The effect of reducing the oscillator spacing*

When the spacing was reduced to 0.01 m as seen in Fig. 10, the most noticeable change is a doubling of the collision rate as 20 steps or collisions now take only 7 s. However the reduction in spacing causes a reduction in collision work or damage by a factor of two as the many small collisions have a lower kinetic energy offering lower damage potential. The damage will of course be minimized when the spacing is reduced to zero and no collisions occur. There is still relative motion on account of the contact stiffness. It would therefore seem best to make buildings as close

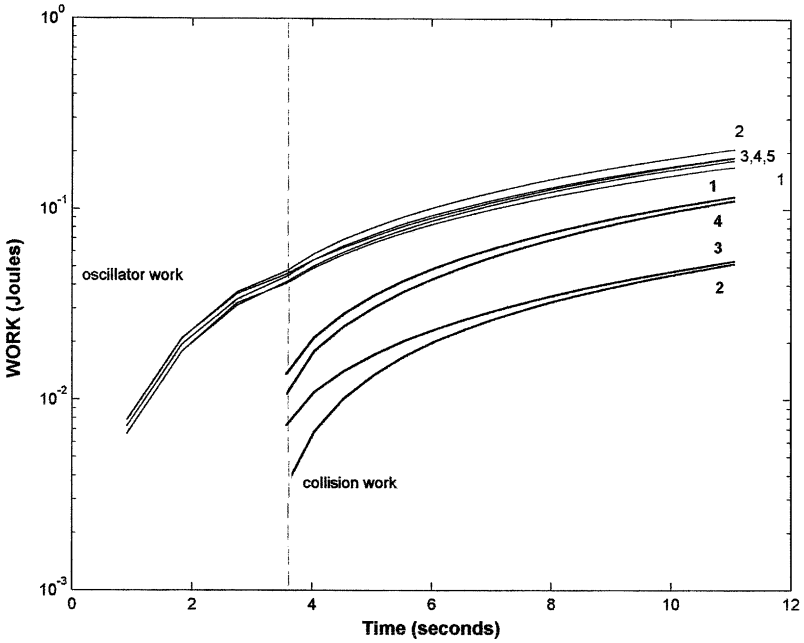

Fig. 9. Oscillator work — and collision work —, separation 0.1 m, contact stiffness 100(1 + i)N/m.

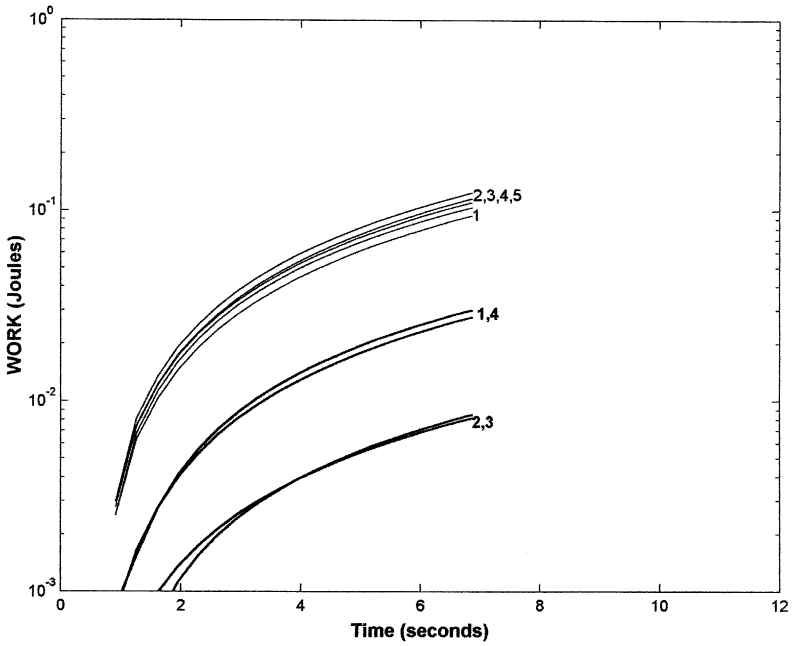

Fig. 10. Oscillator work — and collision work —, separation 0.01 m, contact stiffness 100(1 + i)N/m.

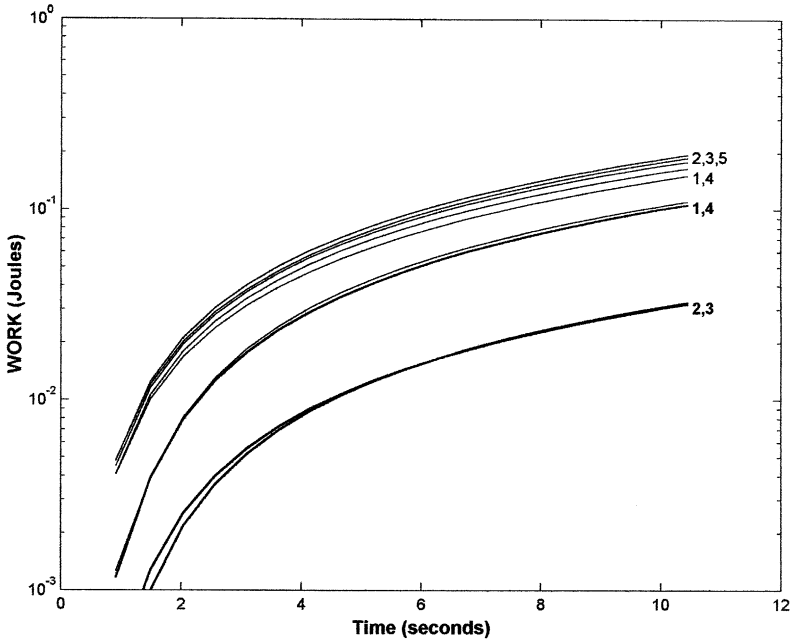

Fig. 11. Oscillator work — and collision work —, separation 0.01 m, contact stiffness 20(1 + i)N/m.

as possible, or fill the gap with a compliant medium to reduce damage. A further benefit of filling the gap between real buildings is that further damping of the in-phase motion would occur in the shearing action between the parallel surfaces.

5.3. Reduction of the contact stiffness

Fig. 11 shows the work in the collisions and in the oscillators when the conditions of the previous test were repeated but the contact stiffness was reduced to 20 N/m, close to that of the oscillator stiffness'. The collision rate decreased, or the time for 20 collisions has increased, to the value of the first test, as more time is spent in the softer contact. Accordingly the collision work is now increased causing a slight reduction in oscillator work.

5.4. Reduction of the contact loss factor

The previous test was repeated changing only the contact loss factor to a tenth of the previous value to 0.1. The result seen in Fig. 12 is very similar to Fig. 11 where the loss factor was 1.0. This slightly surprising result was also observed in Ref. [3], from a conventional time-step analysis. The reason is that the total work is independent of the damping of the oscillators, and up to half of the oscillator energy can be lost in a collision. Therefore provided the contact damping is at least as great as the oscillator damping there will be little influence on the distribution of energy loss. These observations also suggest that for this type of energy analysis the description of contact damping need not be very precise.

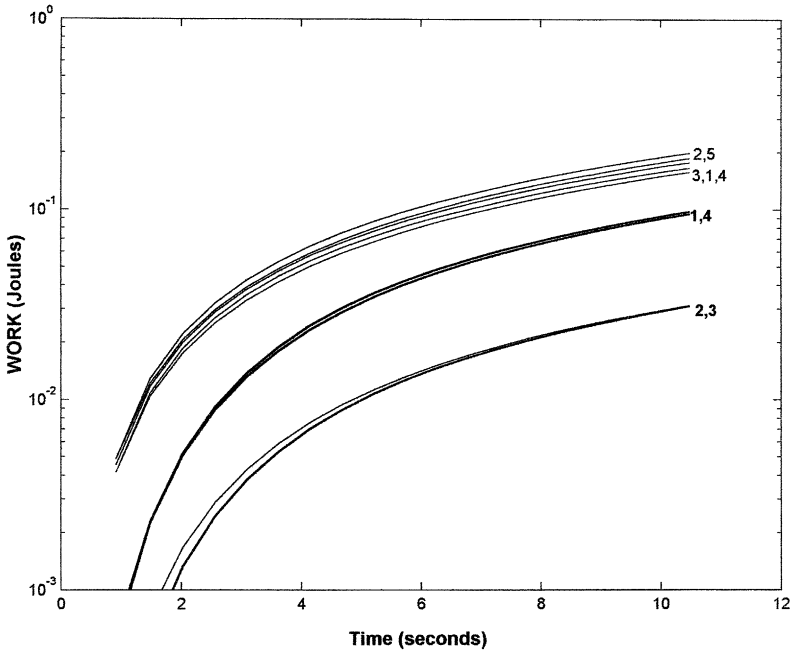

Fig. 12. Oscillator work — and collision work —, separation 0.01 m, contact stiffness $20(1+i0.1)$N/m.

6. Conclusions

A time-stepping power balance program has been made to calculate the work due to oscillator internal hysteresis and the work of collision between a series of colliding oscillators. The program appears to be very stable, which is to be expected from a first order differential equation. Time steps are made at the mean collision rate for a single oscillator.

In this analysis a broadband random excitation was used giving input power and hence dissipation that is only determined by the excitation level and the mass of the oscillators. The total dissipation is therefore insensitive to the loss factors of the contact and oscillators.

The energy exchange between oscillators in a collision was found to be almost proportional to the oscillator energy difference. The energy dissipation in collision is proportional to the square of relative velocity at the instant of impact.

The most important parameter for collision damage appears to be the spacing between oscillators. As the spacing is increased the collision rate decreases, but the work of collision increases because the collision is more violent with increased freedom of movement. The greatest damage is between the outer oscillator pairs for the same reason, i.e. greater freedom leads to more damage.

The damping within the contact has little influence on the proportion of damage provided the loss factor is greater than the oscillator damping.

The greatest quantity of damage occurs in the collision if the contact stiffness is comparable with the oscillator stiffness. This effect could be used to advantage if a damped compliant in-fill was used between buildings to absorb energy from out-of-phase motion. On average the

out-of-phase motion is only half of the total available energy, the other half is in-phase motion which is not influenced in the collision. It may be possible that a compliant in-fill could also absorb energy from this type of motion by a shearing action.

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Appendix A. Hysteretic damping work for harmonic excitation and impulses

The objective here is to summarize some properties of a hysteretic damping model that can be used to linearize a non-linear spring characteristic. The first application is for the coefficient of restitution of damped springs. The second is to state the division of energy dissipation within the contact and oscillator during the contact duration.

Loss factor η_r is useful because it directly describes the work done D_r per harmonic loading cycle irrespective of linearity, as seen in Fig. 3 and Eq. (3b):

$$\eta_r = D_r/(2\pi S_{r\,\max}), \tag{A.1}$$

where $S_{r\,\max}$ is the peak strain energy in element r. For a single impact, where there is approximately a quarter a cycle of contact, and the displacement begins from zero as in Fig. 3, the work $\hat{D}_r$ is one quarter of this:

$$\hat{D}_r = \eta_r \pi S_{r\,\max}/2. \tag{A.2}$$

However it should be noted that if the material had velocity dependent viscous damping a single impact is half a cycle or half sine wave and the work would be half of Eq. (A.1):

$$\hat{D}_r = \eta_r \pi S_{r\,\max}. \tag{A.3}$$

If a complex interaction stiffness is employed $\bar{k}_{ir} = k_{ir}(1 + \mathrm{i}\eta_{ir})$, and complex contact stiffness $k_{cr} = k'_{cr}(1 + \mathrm{i}\eta_{cr})$ The complex interaction frequency in contact equation (38) becomes

$$\hat{\omega}_{ir} = \pm\sqrt{\frac{k_{ir} + k'_{cr} + \mathrm{i}(\eta_{ir} + \eta_{cr})}{m_{ir}}} \tag{A.4}$$

and rewriting Eq. (A.4) as

$$\hat{\omega}_{ir} = \sqrt{\frac{(k_{ir} + k'_{cr})\,(1 + \mathrm{i}\eta_{cr})}{m_{ir}}}. \tag{A.5}$$

The imaginary component of the above expression could for light damping be expanded as

$$\mathrm{Im}(\hat{\omega}_{ir}) = \frac{\eta_r}{2}\mathrm{Re}(\hat{\omega}_{ir}). \tag{A.6}$$

This expression assumes viscous velocity dependent damping, giving the energy loss for an impulse in Eq. (A.3). For hysteretic, amplitude dependent damping half of this value must be used to obtain the correct work given in Eq. (A.2), i.e.

$$\mathrm{Im}(\hat{\omega}_{\mathrm{i}r}) = \frac{\eta_r}{4}\mathrm{Re}(\hat{\omega}_{\mathrm{i}r}). \tag{A.7}$$

These employ a total loss factor η_r that describes the energy loss during contact within both the contact $\hat{D}_{\mathrm{c}r}$ and in the oscillator $\hat{D}_{\mathrm{i}r}$, where the total loss in contact $\hat{D}_r = \hat{D}_{\mathrm{i}r} + \hat{D}_{\mathrm{c}r}$:

$$\eta_r = \frac{(\eta_{\mathrm{i}r}k_{\mathrm{i}r} + \eta_{\mathrm{c}r}k'_{\mathrm{c}r})}{(k_{\mathrm{i}r} + k'_{\mathrm{c}r})}.$$

The proportion of energy loss in the contact is

$$\frac{\hat{D}_{\mathrm{c}r}}{\hat{D}_r} = \frac{\eta_{\mathrm{c}r}k'_{\mathrm{c}r}}{(k_{\mathrm{i}r} + k'_{\mathrm{c}r})} \tag{A.8}$$

while the loss within the oscillator in the same time interval is

$$\frac{\hat{D}_{\mathrm{i}r}}{\hat{D}_r} = \frac{\eta_{\mathrm{i}r}k'_{\mathrm{i}r}}{(k_{\mathrm{i}r} + k'_{\mathrm{c}r})}. \tag{A.9}$$

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